ECE 598: The Speech Chain

Lecture 4: Sound
Today

- Ideal Gas Law + Newton’s Second = Sound
- Forward-Going and Backward-Going Waves
- Pressure, Velocity, and Volume Velocity
- Boundary Conditions:
  - Open tube: zero pressure
  - Closed tube: zero velocity
- Resonant Frequencies of a Pipe
Speech Units: cgs

- **Distance measured in cm**
  - 1 cm = length of vocal folds; 15-18 cm = length vocal tract
- **Mass measured in grams (g)**
  - 1 g = mass of one cm$^3$ of H$_2$O or biological tissue
  - 1 g = mass of the vocal folds
- **Time measured in seconds (s)**
- **Volume measured in liters (1 L = 1000 cm$^3$)**
  - 1 L/s = air flow rate during speech
- **Force measured in dynes (1 d = 1 g cm/s$^2$)**
  - 1000 dynes = force of gravity on 1 g (1 cm$^3$) of H$_2$O
- **Pressure measured in dynes/cm$^2$ or cm H$_2$O**
  - 1000 d/cm$^2$ = pressure of 1 cm H$_2$O
  - Lung pressure varies from 1-10 cm H$_2$O
Consider three blocks of air in a pipe.

- Boundaries are at $x$ (cm) and $x+dx$ (cm).
- Pipe cross-sectional area = $A$ (cm$^2$)
- Volume of each block of air: $V = A \, dx$ (cm$^3$)
- Mass of each block of air = $m$ (grams)
- Density of each block of air: $\rho = m/(Adx)$ (g/cm$^3$)
Step 1: Middle Block Squished

- Velocity of air is $v$ or $v + dv$ (cm/s)
  - In the example above, $dv$ is a negative number!!!
- In $dt$ seconds, Volume of middle block changes by
  
  $$dV = A \left( (v + dv) - dv \right) dt = A \, dv \, dt$$
  
  $$cm^3 = (cm^2) \, (cm/s) \, s$$
- Density of middle block changes by
  
  $$d\rho = m/(V+\Delta V) - m/V$$
  
  $$\approx -\rho \, dV/V = -\rho \, dt \, dv/dx$$
- Rate of Change of the density of the middle block
  
  $$d\rho/dt = -\rho \, dv/dx$$
  
  $$(g/cm^3)/s = (g/cm^3) \, (m/s) / m$$
Step 2: The Pressure Rises

- Ideal Gas Law: \( p = \rho RT \) (pressure proportional to density, temperature, and a constant \( R \))
- Adiabatic Ideal Gas Law: \( \frac{dp}{dt} = c^2 \frac{d\rho}{dt} \)
  - When gas is compressed quickly, “\( T \)” and “\( \rho \)” both increase
  - This is called “adiabatic expansion” --- it means that \( c^2 > R \)
  - “\( c \)” is the speed of sound!!
    - “\( c \)” depends on chemical composition (air vs. helium), temperature (body temp. vs. room temp.), and atmospheric pressure (sea level vs. Himalayas)
- \( \frac{dp}{dt} = c^2 \frac{d\rho}{dt} = -\rho c^2 \frac{dv}{dx} \)
Step 3: Pressure X Area = Force

- Force acting on the air between 😞 and 😊:
  \[ F = pA - (p+dp)A = -A \, dp \]
  dynes = \( \text{(cm}^2\) \( \text{(d/cm}^2\) \)
Step 4: Force Accelerates Air

Air velocity has changed here!

- Newton’s second law:
  \[ F = m \frac{dv}{dt} = (\rho A dx) \frac{dv}{dx} \]
  \[-dp/dx = \rho \frac{dv}{dt} \]
  \[(d/cm^2)/cm = (g/cm^3) (cm/s)/s\]
Acoustic Constitutive Equations

- **Newton’s Second Law:**
  \[-\frac{dp}{dx} = \rho \frac{dv}{dt}\]

- **Adiabatic Ideal Gas Law:**
  \[\frac{dp}{dt} = -\rho c^2 \frac{dv}{dx}\]

- **Acoustic Wave Equation:**
  \[\frac{d^2p}{dt^2} = c^2 \frac{d^2p}{dx^2}\]
  \[\text{pressure/s}^2 = (\text{cm/s})^2 \text{ pressure/cm}^2\]

**Things to notice:**
- p must be a function of both time and space: p(x,t)
- “c” is a speed (35400 cm/s, the speed of sound at body temperature, or 34000 cm/s at room temperature)
- “ct” is a distance (distance traveled by sound in t seconds)
Solution: Forward and Backward Traveling Waves

Wave Number $k$:

$$k = \frac{\omega}{c}$$

(radians/cm) = (radians/sec) / (cm/sec)

Forward-Traveling Wave:

$$p(x,t) = p_+ e^{i(\omega t - kx)} = p_+ e^{i(\omega(t-x/c))}$$

$$\frac{d^2p}{dt^2} = c^2 \frac{d^2p}{dx^2}$$

$$-\omega^2 p_+ = -(kc)^2 p_+$$

Backward-Traveling Wave:

$$p(x,t) = p_- e^{i(\omega t + kx)} = p_- e^{i(\omega(t+x/c))}$$

$$\frac{d^2p}{dt^2} = c^2 \frac{d^2p}{dx^2}$$

$$-\omega^2 p_- = -(kc)^2 p_-$$
Forward and Backward Traveling Waves

\[
\cos(kx-t\omega) \text{ at } t=0, \quad \omega=400\pi
\]

\[
\cos(kx+t\omega) \text{ at } t=0, \quad \omega=400\pi
\]

\[
\cos(kx-t\omega) \text{ at } t=1\text{ms}
\]

\[
\cos(kx+t\omega) \text{ at } t=1\text{ms}
\]
Other Wave Quantities Worth Knowing

- **Wave Number** $k$:
  \[ k = \frac{\omega}{c} \]
  
  (radians/cm) = (radians/sec) / (cm/sec)

- **Wavelength** $\lambda$:
  \[ \lambda = \frac{c}{f} = \frac{2\pi}{k} \]
  
  (cm/cycle) = (cm/sec) / (cycles/sec)

- **Period** $T$:
  \[ T = \frac{1}{f} \]
  
  (seconds/cycle) = 1 / (cycles/sec)
Air Particle Velocity

- **Forward-Traveling Wave**
  \[ p(x,t) = p_+ e^{j(\omega(t-x/c))}, \ v(x,t) = v_+ e^{j(\omega(t-x/c))} \]

- **Backward-Traveling Wave**
  \[ p(x,t) = p_- e^{j(\omega(t+x/c))}, \ v(x,t) = v_- e^{j(\omega(t+x/c))} \]

- **Newton’s Second Law:**
  \[
  -\frac{dp}{dx} = \rho \frac{dv}{dt} \\
  (\omega/c)p_+ = \omega \rho v_+ \\
  -(\omega/c)p_- = \omega \rho v_- 
  \]

- **Characteristic Impedance of Air:** \( z_0 = \rho c \)
  \[ v_+ = \frac{p_+}{\rho c} \]
  \[ v_- = -\frac{p_-}{\rho c} \]
Volume Velocity

- In a pipe, it sometimes makes more sense to talk about movement of all of the molecules at position $x$, all at once.
  - $A(x) = \text{cross-sectional area of the pipe at position } x$ (cm$^2$)
  - $v(x,t) = \text{velocity of air molecules (cm/s)}$
  - $u(x,t) = A(x)v(x,t) = \text{“volume velocity” (cm}^3/\text{s} = \text{mL/s)}$
Acoustic Waves

- **Pressure**
  \[ p(x,t) = e^{j\omega t} (p_+ e^{-jkx} + p_- e^{jkx}) \]

- **Velocity**
  \[ v(x,t) = e^{j\omega t} \left( \frac{1}{\rho c} (p_+ e^{-jkx} - p_- e^{jkx}) \right) \]
Standing Waves: Resonances in the Vocal Tract
Boundary Conditions

Hard Wall Boundary Suppose the tube is closed at $x = 0$ by a hard wall. Air cannot travel through a hard wall, therefore:

$$U(0, j\Omega) = 0$$

but

$$U(0, j\Omega) = \frac{1}{\rho c} (P_+(j\Omega) - P_-(j\Omega))$$

Therefore

$$P_+(j\Omega) = P_-(j\Omega)$$

Open Space Boundary Suppose that the tube is open to the room at $x = L$. Air flow from the end of a small, open-ended tube will never change the ambient pressure outside the tube, therefore

$$P_{tot}(L, j\Omega) = P_0 + P(L, j\Omega) = P_0$$

$$P(L, j\Omega) = 0$$
Boundary Conditions

- **Pressure**=0 at x=L
  \[ 0 = p(L,t) = e^{j\omega t} (p_+ e^{-jkL} + p_- e^{jkL}) \]
  \[ 0 = p_+ e^{-jkL} + p_- e^{jkL} \]

- **Velocity**=0 at x=0
  \[ 0 = v(0,t) = e^{j\omega t} \left(\frac{1}{\rho c} (p_+ e^{-jk0} - p_- e^{jk0})\right) \]
  \[ 0 = p_+ e^{-jk0} - p_- e^{jk0} \]
  \[ 0 = p_+ - p_- \]
  \[ p_+ = p_- \]
Two Equations in Two Unknowns

(p_+ and p_-)

- Two Equations in Two Unknowns
  \[0 = p_+ e^{-jkL} + p_- e^{jkL}\]
  \[p_+ = p_-\]

- Combine by Variable Substitution:
  \[0 = p_+ (e^{-jkL} + e^{jkL})\]
...and now, more useful trigonometry...

- **Definition of complex exponential:**
  \[ e^{jkL} = \cos(kL) + j \sin(kL) \]
  \[ e^{-jkL} = \cos(-kL) + j \sin(-kL) \]

- **Cosine is symmetric, Sine antisymmetric:**
  \[ e^{-jkL} = \cos(kL) - j \sin(kL) \]

- **Re-combine to get useful equalities:**
  \[ \cos(kL) = 0.5(e^{jkL} + e^{-jkL}) \]
  \[ \sin(kL) = -0.5 j (e^{jkL} - e^{-jkL}) \]
Two Equations in Two Unknowns

\[ 0 = p_+ e^{-jkL} + p_- e^{jkL} \]

\[ p_+ = p_- \]

Combine by Variable Substitution:

\[ 0 = p_+ (e^{-jkL} + e^{jkL}) \]

\[ 0 = 2p_+ \cos(kL) \]

Two Possible Solutions:

\[ 0 = p_+ \ (\text{Meaning, amplitude of the wave is 0}) \]

\[ 0 = \cos(kL) \ (\text{Meaning...}) \]
Resonant Frequencies of a Uniform Tube, Closed at One End, Open at the Other End

- $p_+$ can only be nonzero (the amplitude of the wave can only be nonzero) at frequencies that satisfy...

\[
0 = \cos(kL)
\]

\[
kL = p/2, 3p/2, 5p/2, \ldots
\]

\[
k_1 = \pi/2L, \ \omega_1 = \pi c/2L, \ F_1 = c/4L
\]

\[
k_2 = 3\pi/2L, \ \omega_2 = 3\pi c/2L, \ F_2 = 3c/4L
\]

\[
k_3 = 5\pi/2L, \ \omega_3 = 5\pi c/2L, \ F_3 = 5c/4L
\]

...
Resonant Frequencies of a Uniform Tube, Closed at One End, Open at the Other End

For example, if the vocal tract is 17.5cm long, and the speed of sound is 350m/s at body temperature, then \( \frac{c}{L} = 2000\text{Hz} \), and

\[
F_1 = 500\text{Hz} \\
F_2 = 1500\text{Hz} \\
F_3 = 2500\text{Hz}
\]
Closed at One End, Open at the Other End = “Quarter-Wave Resonator”

- The wavelength are:
  \[ \lambda_1 = 4L \]
  \[ \lambda_2 = \frac{4L}{3} \]
  \[ \lambda_3 = \frac{4L}{5} \]
The pressure and velocity are

\[ p(x,t) = p_+ e^{j\omega t} (e^{-jkx} + e^{jkx}) \]
\[ v(x,t) = e^{j\omega t} \left( \frac{p_+}{\rho c} \right) (e^{-jkx} - e^{jkx}) \]

Remember that \( p_+ \) encodes both the magnitude and phase, like this:

\[ p_+ = P_+ e^{j\phi} \]
\[ \text{Re}\{p(x,t)\} = 2P_+ \cos(\omega t + \phi) \cos(kx) \]
\[ \text{Re}\{v(x,t)\} = \left( \frac{2P_+}{\rho c} \right) \cos(\omega t + \phi) \sin(kx) \]
The “standing wave pattern” is the part that doesn’t depend on time:

\[ |p(x)| = 2P_+ \cos(kx) \]

\[ |v(x)| = \frac{2P_+}{\rho c} \sin(kx) \]
Standing Wave Patterns: Quarter-Wave Resonators

P(x) Standing Wave, F1

P(x) Standing Wave, F2

P(x) Standing Wave, F3

U(x) Standing Wave, F1

U(x) Standing Wave, F2

U(x) Standing Wave, F3

Glotis

Lips
Half-Wave Resonators

- If the tube is open at $x=0$ and at $x=L$, then boundary conditions are $p(0,t)=0$ and $p(L,t)=0$.
- If the tube is closed at $x=0$ and at $x=L$, then boundary conditions are $v(0,t)=0$ and $v(L,t)=0$.
- In either case, the resonances are:
  \[ F_1 = 0, \quad F_2 = c/2L, \quad F_3 = c/L, \quad F_3 = 3c/2L \]
- Example, vocal tract closed at both ends:
  \[ F_1 = 0, \quad F_2 = 1000\text{Hz}, \quad F_3 = 2000\text{Hz}, \quad F_3 = 3000\text{Hz} \]
Standing Wave Patterns: Half-Wave Resonators

- Tube Closed at Both Ends
- Tube Open at Both Ends
Schwa and Invv
(the vowels in “a tug”)

F1 = 500 Hz = c/4L
F2 = 1500 Hz = 3c/4L
F3 = 2500 Hz = 5c/4L
Front Cavity Resonances of a Fricative

/s/: Front Cavity Resonance = 4500Hz
4500Hz = c/4L  if
Front Cavity Length is L=1.9cm

/sh/: Front Cavity Resonance = 2200Hz
2200Hz = c/4L  if
Front Cavity Length is L=4.0cm
Summary

- Newton’s Second Law:
  \[-\frac{dp}{dx} = \rho \frac{dv}{dt}\]

- Adiabatic Ideal Gas Law:
  \[\frac{dp}{dt} = -\rho c^2 \frac{dv}{dx}\]

- Acoustic Wave Equation:
  \[\frac{d^2p}{dt^2} = c^2 \frac{d^2p}{dx^2}\]

- Solutions
  \[p(x,t) = e^{j\omega t} (p_+ e^{-jkx} + p_- e^{jkx})\]
  \[v(x,t) = e^{j\omega t} \left(\frac{1}{\rho c}(p_+ e^{-jkx} - p_- e^{jkx})\right)\]

- Resonances
  - Quarter-wave resonator: \(F_1 = c/4L, F_2 = 3c/4L, F_3 = 5c/4L\)
  - Half-wave resonator: \(F_1 = 0, F_2 = c/2L, F_3 = c/L\)

- Standing-wave Patterns
  \[|p(x)| = 2P_+ \cos(kx)\]
  \[|v(x)| = \left(2P_+/\rho c\right) \sin(kx)\]