Line spectral frequencies are poles and zeros of the glottal driving-point impedance of a discrete matched-impedance vocal tract model

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This correspondence demonstrates that the line spectral frequencies (LSFs) are the pole and zero frequencies of the glottal driving-point impedance of a discrete matched-impedance vocal-tract model. Several well-known characteristics of the LSFs, including the interlacing of pole and zero frequencies, are shown to follow naturally from this proof. © *2000 Acoustical Society of America.* [S0001-4966(00)02307-9]

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INTRODUCTION

Linear prediction analysis is often used to calculate the cross-sectional area function of a particular discrete, linear, matched-impedance vocal-tract model. The discrete matched-impedance model (DMI model) is a concatenated tube model, of a length specified by the experimenter, in which all terminations are lossless except for a single matched-impedance termination at the glottis (Furui, 1989). The DMI model is useful primarily because, given an all-pole estimate of the vocal-tract transfer function, the cross-sectional areas of tube sections in the DMI model are unique, and may be calculated using an efficient algorithm. The model is often used in speech coding, synthesis, and recognition systems whose designers wish to simulate speech production without the cost of calculating a more accurate area function.

The purpose of this letter is to demonstrate that the line spectral frequencies (LSFs) are the poles and zeros of the glottal driving-point impedance of the standard DMI model. This new definition of the LSFs is more concise than the standard definition, according to which the LSFs are the resonances of two variants of the DMI model (Furui, 1989). The new definition also allows one to think about the LSFs using a type of physical reasoning not previously applicable. For example, many acousticians are aware of Foster's reactance theorem, which states that the poles and zeros of a driving-point reactance alternate in frequency (Foster, 1924). Section III uses Foster's reactance theorem to demonstrate a concise alternative proof of the well-known and useful interlacing property of the LSFs (Sugamura and Itakura, 1981; Soong and Juang, 1984).

This letter is organized as follows. Section I reviews the definitions and mathematical properties of the DMI model and of the LSFs. Section II proves that the glottal driving-point impedance of the DMI model is a rational function of the discrete-time transform variable z. Section III proves that the LSFs are the pole and zero frequencies of the driving-point impedance, and that the vocal-tract transfer function is stable if and only if the poles and zeros alternate in frequency.

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I. BACKGROUND

During the production of oral vowels and glides, the vocal tract may be modeled as a one-dimensional acoustic resonator excited by a high-impedance volume velocity source located at the glottis (O'Shaughnessy, 1987). The shape of the one-dimensional vocal-tract resonator is commonly approximated using a series of concatenated cylindrical tubes, each of length l, and with varying cross-sectional areas Φ_n, as shown in Fig. 1 (the derivations in this article will assume an arbitrary number of tube sections; for convenience, Fig. 1 shows only five tubes). Let $\tilde{U}_n^+(s)$ denote the forward-going volume velocity wave at the left end of the nth such tube, and let $-\tilde{U}_n^-(s)$ denote the backward-going wave at the right end of the same tube, where $s=\sigma+j\Omega$ is the complex frequency variable. The volume velocity at the left end of the nth such tube, $\tilde{U}(-nl,s)$, is

$$\tilde{U}(-nl,s)=\tilde{U}_n^+(s)-e^{-sl/c}\tilde{U}_n^-(s), \tag{1}$$

and the acoustic pressure at the same point is

$$\tilde{P}_n(s)=Z_n[\tilde{U}_n^+(s)+e^{-sl/c}\tilde{U}_n^-(s)], \quad Z_n\equiv\frac{\rho c}{\Phi_n}, \tag{2}$$

where ρ is the density of air, c is the speed of sound, and Z_n is called the characteristic impedance of the nth tube section. By applying pressure and volume velocity continuity constraints at the boundary between tubes n and $n+1$, it is possible to define a constant reflection coefficient k_n such that

$$\tilde{U}_n^+(s)=e^{-sl/c}[(1-k_n)\tilde{U}_{n+1}^+(s)-k_n\tilde{U}_n^-(s)], \tag{3}$$

$$\tilde{U}_{n+1}^-(s)=e^{-sl/c}[(1+k_n)\tilde{U}_n^-(s)+k_n\tilde{U}_{n+1}^+(s)], \tag{4}$$

where

$$k_n=\frac{Z_n-Z_{n+1}}{Z_n+Z_{n+1}}. \tag{5}$$

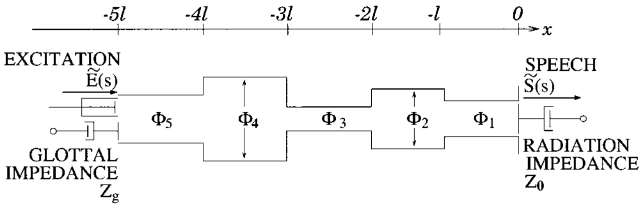

FIG. 1. Concatenated tube model of the vocal tract.

The vocal-tract model is terminated at the glottis by a massless, frictionless piston driven by signal $\tilde{E}(s)$, in parallel with a passive termination element with impedance Z_g. If Z_g is independent of frequency, the termination can be written using a constant reflection coefficient k_g

$$\tilde{U}_N^+(s)=\tilde{E}(s)-k_g\tilde{U}_N^-(s)e^{-sl/c}, \tag{6}$$

where N is the number of tube sections (e.g., Fig. 1 shows the case in which $N=5$). The vocal tract is terminated at the lips by radiation impedance Z_0, which yields reflection coefficient k_0

$$\tilde{U}_1^-(s)=k_0\tilde{U}_1^+(s)e^{-sl/c}. \tag{7}$$

Inspection of Eqs. (3) through (7) reveals that, in an undriven system, $\tilde{U}_n^+(s)$ and $\tilde{U}_n^-(s)$ are periodic in s

$$\tilde{U}_n^+(s)=\tilde{U}_n^+(s\pm j2\pi mF_s),\quad F_s\equiv\frac{c}{2l},\ m=1,2,\ldots, \tag{8}$$

and the same relation holds for $\tilde{U}_n^-(s)$. Without loss of information, we may therefore define discrete-time transform quantities $\tilde{U}_n^+(z)$, $\tilde{U}_n^-(z)$, $P(n,z)$, and $U(n,z)$. Furui (1989) describes a number of different discrete-time implementations of Fig. 1; one of the most common implementations is the lattice filter structure shown in Fig. 2. For convenience of implementation, the relationship between the discrete-time quantities in Fig. 2 and the corresponding continuous-time quantities in Fig. 1 may be defined to include a constant delay and a constant scaling factor; thus,

$$U_n^+(z)=\tilde{G}_n(s)\tilde{U}_n^+(s)|_{s=F_s\log z}, \tag{9}$$

$$U_n^-(z)=e^{-s/2F_s}\tilde{G}_n(s)\tilde{U}_n^-(s)|_{s=F_s\log z}, \tag{10}$$

$$P(n,z)=\tilde{G}_n(s)\tilde{P}(-nl,s)|_{s=F_s\log z}, \tag{11}$$

$$U(n,z)=\tilde{G}_n(s)\tilde{U}(-nl,s)|_{s=F_s\log z}, \tag{12}$$

where

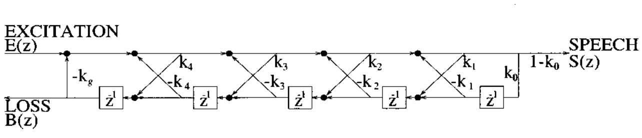

FIG. 2. Discrete-time implementation of a concatenated tube model of the vocal tract.

$$\tilde{G}_n(s)\equiv e^{-ns/2F_s}\prod_{i=1}^{n-1}(1-k_i). \tag{13}$$

The ''speech signal'' may be conveniently defined to be the volume velocity which passes through the radiation impedance element Z_0, i.e.,

$$\tilde{S}(s)=(1-k_0)\tilde{U}+_1^+(s)e^{-sl/c}. \tag{14}$$

The transfer function of the vocal tract may be defined as the ratio of $\tilde{S}(s)$ to the excitation signal $\tilde{E}(s)$. During oral vowels and glides, the transfer function is an all-pole function, with the form

$$\tilde{T}(s)=\frac{\tilde{S}(s)}{\tilde{E}(s)}=\frac{1}{\tilde{A}(s)}=\prod_{m=1}^{\infty}\frac{s_m s_m^*}{(s-s_m)(s-s_m^*)}. \tag{15}$$

The discrete-time transfer function is likewise defined to be

$$T(z)=\frac{S(z)}{E(z)}=\frac{1}{A(z)}. \tag{16}$$

The roots of $\tilde{A}(s)$, $s_m=\pi B_m+j2\pi F_m$, are the resonant frequencies of the vocal tract; the roots of $A(z)$ are $z_m=e^{s_m/F_s}$. This section considers two standard methods for calculating the resonant frequencies: the series impedance method, and the Levinson–Durbin recursion.

One of the most general methods for calculating s_m requires cutting the vocal-tract resonator in half at some coordinate x_n, creating two half-resonators called the back cavity $(x<x_n)$ and the front cavity $(x>x_n)$. The geometry of the back cavity dictates a relationship between pressure $\tilde{P}(x_n,s)$ and flow $\tilde{U}(x_n,s)$, and the geometry of the front cavity dictates a second relationship. If these relationships are linear, they may be expressed as a back-cavity impedance $\tilde{Z}_b(x_n,s)$ and a front-cavity impedance $\tilde{Z}_f(x_n,s)$

$$\tilde{P}(x_n,s)=-\tilde{Z}_b(x_n,s)\tilde{U}(x_n,s), \tag{17}$$

$$\tilde{P}(x_n,s)=\tilde{Z}_f(x_n,s)\tilde{U}(x_n,s). \tag{18}$$

If x_n is defined to be the left end of the nth tube section, $x_n\equiv-nl$, then Eqs. (17) and (18) may also be written using discrete-time impedances

$$\begin{aligned}P(n,z)&=-Z_b(n,z)U(n,z),\\ Z_b(n,z)&\equiv\tilde{Z}_b(-nl,s)|_{s=F_s\log z},\end{aligned} \tag{19}$$

$$\begin{aligned}P(n,z)&=Z_f(n,z)U(n,z),\\ Z_f(n,z)&\equiv\tilde{Z}_f(-nl,s)|_{s=F_s\log z}.\end{aligned} \tag{20}$$

The pressure and flow at $x_n=-nl$ must satisfy Eqs. (19) and (20) simultaneously. Solving, we find that $U(n,z)$ can only be nonzero at frequencies for which

$$Z_b(n,z)+Z_f(n,z)=0, \tag{21}$$

while $P(n,z)$ can only be nonzero at frequencies for which

$$Y_b(n,z)+Y_f(n,z)=0, \tag{22}$$

where $Y_b=1/Z_b$ and $Y_f=1/Z_f$. Barring any pole-zero cancellations, it can be shown that the solutions to both Eqs.

(21) and (22) are given by the resonant frequencies z_m. Equation (21) is sometimes called the ''series impedance'' method of calculating resonant frequencies, and Eq. (22) is called the ''parallel admittance'' method.

The series impedance method may be used for any vocal-tract geometry, and with any assumed distribution of losses. The Levinson–Durbin recursion is much more computationally efficient than the series impedance method, but it has only been shown to be useful for two particular distributions of losses. Specifically, the Levinson–Durbin recursion assumes that all of the vocal-tract boundaries are lossless except for a single matched-impedance termination, which may be placed at either the lips or the glottis (Furui, 1989). For convenience, this article uses the term ''DMI model'' to mean a vocal-tract model which is lossless except for a single matched-impedance termination at the glottis. In the DMI model, the radiation impedance and glottal impedance are given by

$$Z_0=0, \quad Z_g=Z_N, \tag{23}$$

where Z_N is as given in Eq. (2). The corresponding reflection coefficients are

$$k_0=-1, \quad k_g=0. \tag{24}$$

With this matched-impedance termination, the energy lost by the vocal tract is

$$\text{loss spectrum}=\tfrac{1}{2}Z_g|B(z)|^2, \tag{25}$$

where

$$B(z)\equiv U_N^-(z)=z^{-N}A(z^{-1})S(z). \tag{26}$$

Further details of the Levinson–Durbin recursion are given elsewhere (O'Shaughnessy, 1987; Furui, 1989); to understand the derivations in this article, it is sufficient to know that the algorithm results in the distribution of losses specified by Eqs. (23)–(26).

The line spectral frequencies (LSFs) are typically defined as the resonant frequencies of two variants of the DMI model (Furui, 1989). Both modified models are created by first calculating the cross-sectional areas Φ_n of the DMI model using the Levinson–Durbin recursion, and then creating a new model which has the same Φ_n, and the same lossless termination at the lips, but a different glottal termination.

In the first modified model, which we may call the ''Q model,'' the glottal impedance is set to $Z_g=0$. From Eq. (5) a glottal impedance of $Z_g=0$ results in a glottal reflection coefficient of $k_g=1$. Combining information from Eqs. (6), (16), and (26), the forward-going wave at the left end of the N tube is shown to be

$$U_N^+(z)=E(z)-B(z)=Q(z)S(z), \tag{27}$$

$$Q(z)\equiv A(z)-z^{-N}A(z^{-1}). \tag{28}$$

The resonances of this system are the frequencies at which $S(z)$ can be nonzero even when $U_N^+(z)$ is zero. This condition is met at frequencies q_i for which $Q(e^{jq_i})=0$. $Q(z)$ can be factored as

$$Q(z)=\prod_{i=0}^{N-1}(1-e^{jq_i}z^{-1}), \tag{29}$$

where the frequencies q_i are real numbers, $0\leqslant q_i<2\pi$, and satisfy the following symmetry condition:

$$q_i=\begin{cases}0 & i=0\\ 2\pi-q(N-i) & 1\leqslant i\leqslant N-1.\end{cases} \tag{30}$$

In the second modified model, which we may call the ''R model,'' the glottal impedance is set to $Z_g=\infty$. From Eq. (5), a glottal impedance of $Z_g=\infty$ results in a glottal reflection coefficient of $k_g=-1$; thus,

$$U_N^+(z)=E(z)+B(z)=R(z)S(z), \tag{31}$$

$$R(z)\equiv A(z)+z^{-N}A(z^{-1}). \tag{32}$$

The resonances of this system are the real frequencies r_i at which $R(e^{jr_i})=0$. $R(z)$ can be factored as

$$R(z)=\prod_{i=0}^{N-1}(1-e^{jr_i}z^{-1}), \tag{33}$$

where the frequencies r_i satisfy

$$r_i=2\pi-r_{(N-1-i)}, \quad 0\leqslant i\leqslant N-1. \tag{34}$$

The line spectral frequencies q_i and r_i are a unique and invertible representation of the polynomial $A(z)$. They are real, and, if and only if $1/A(z)$ is stable, they are ordered according to the interlacing property (Sugamura and Itakura, 1981; Soong and Juang, 1984)

$$0=q_0<r_0<q_1<\cdots<q_{N-1}<r_{N-1}<2\pi. \tag{35}$$

The interlacing property of the LSFs is useful in speech coding and speech synthesis. A speech-synthesis filter constructed from quantized LSFs (in speech coding), or from LSFs linearly interpolated between two phoneme targets (in speech synthesis), is guaranteed to be stable if and only if the quantized LSFs obey Eq. (35).

II. FORM OF THE GLOTTAL DRIVING-POINT IMPEDANCE

The purpose of this letter is to prove the following theorem.

A. Theorem

The glottal driving-point impedance of the discrete matched-impedance vocal-tract model is given by

$$Z_{dp,g}(z)=-H\frac{Q(z)}{R(z)}, \tag{36}$$

where $Q(z)$ and $R(z)$ are the LSF polynomials given in Eqs. (28) and (32), and H is a real constant.

This section proves that the driving-point impedance is the ratio of two polynomials in z^{-1}, and that the roots of the numerator and denominator alternate. Section III proves that the numerator polynomial is $Q(z)$ and the denominator polynomial is $R(z)$.

Consider applying the series impedance method [Eq. (21)] to the DMI model at the position $x_N=-Nl$ (the left end of the Nth tube section). The ''back cavity'' (the cavity to the left of x_N) has zero length, but it has a finite, real-valued impedance

$$Z_b(N,z)=Z_g\,. \tag{37}$$

The front cavity (the cavity to the right of x_N) consists of the entire vocal tract except the glottal termination, so $Z_f(N,z)$ is appropriately called the glottal driving-point impedance

$$Z_{dp,g}(z)=Z_f(N,z). \tag{38}$$

In the DMI model, all boundaries to the right of the glottis are lossless, so the glottal driving-point impedance is a pure reactance with zero-bandwidth poles and zeros.

In order to prove Eq. (36) it is necessary to invoke Foster's reactance theorem (Foster, 1924). The following restatement of the theorem is based on the analysis of Foster's theorem by Guillemin (1935).

B. Foster's reactance theorem

The most general form of the driving-point impedance of any finite network of positive-valued lossless inductors and capacitors is

$$\tilde{Z}_{dp,M}(j\Omega) = j\tilde{H}\frac{(\Omega^2-\Omega_0^2)(\Omega^2-\Omega_2^2)(\Omega^2-\Omega_4^2)\cdots(\Omega^2-\Omega_{2M-2}^2)}{\Omega(\Omega^2-\Omega_1^2)(\Omega^2-\Omega_3^2)\cdots(\Omega^2-\Omega_{2M-3}^2)}, \tag{39}$$

where $\tilde{H}$ and M are real constants determined by the problem, and

$$0\leqslant\Omega_0<\Omega_1<\Omega_2<\cdots<\Omega_{2M-2}<\infty. \tag{40}$$

Foster's reactance theorem is frequently applied in acoustic phonetic analysis; for example, the analysis of nasals by Fujimura (1962) depends on the alternating property of the poles and zeros as given in Eq. (40). In order to rigorously apply Foster's theorem to the DMI model, however, it is first necessary to equate the DMI model with the limiting case of a finite network of inductors and capacitors.

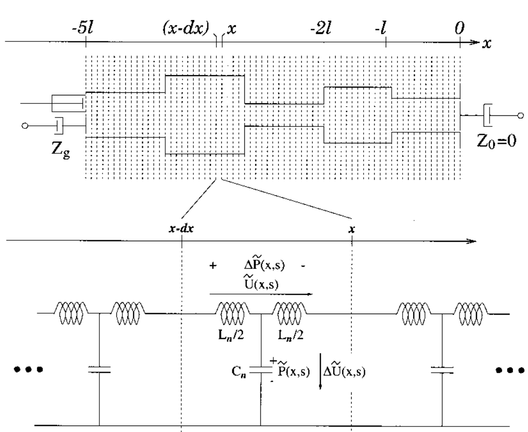

FIG. 3. A continuous hard-walled one-dimensional acoustic resonator, such as the DMI model, may be approximated to arbitrary precision using a discrete ladder network as shown here. $\tilde{U}(x,s)$ is the average of the flow through the two $L_x/2$ elements.

Consider dividing each of the tubes in the DMI model into many smaller tubes, each of length dx, as shown in Fig. 3. If the second and higher derivatives of flow and pressure within each tiny tube section are considered negligible, then each of the tiny tube sections may be approximated using the discrete ladder section shown in Fig. 3. The constitutive equations of the ladder section are

$$\Delta\tilde{U}(x,s)=sC_n\tilde{P}(x,s),\quad C_n\equiv\frac{\Phi_n dx}{\rho c^2},\quad dx\leqslant x+nl\leqslant l, \tag{41}$$

$$\Delta\tilde{P}(x,s)=sL_n\tilde{U}(x,s),\quad L_n\equiv\frac{\rho dx}{\Phi_n},\quad dx\leqslant x+nl\leqslant l, \tag{42}$$

As the number of ladder sections approaches infinity, dx approaches zero, and Eqs. (41) and (42) approach the acoustic constitutive equations of the DMI model. Guillemin (1935) shows that the driving-point impedance of the network shown in Fig. 3 is of the form given in Eq. (39), with M equal to the number of ladder sections, and where $Z_0=0$ implies $\Omega_0=0$. The driving-point impedance of the DMI model is therefore

$$\tilde{Z}_{dp,g}(j\Omega)=\lim_{M\to\infty}\tilde{Z}_{dp,M}(j\Omega),\quad \Omega_0=0. \tag{43}$$

Recall that, because the DMI model is constructed from a discrete number of length-l tubes,

$$\tilde{Z}_{dp,g}(j\Omega)=\tilde{Z}_{dp,g}(j\Omega\pm j2\pi kF_s),\quad k=1,2,\ldots. \tag{44}$$

Equation (44) requires that Ω_{2i} is a zero of $\tilde{Z}_{dp,g}(j\Omega)$ if and only if $\Omega_{2i}\pm 2\pi kF_s$ is also a zero. As M approaches infinity, therefore, the poles and zeros of $\tilde{Z}_{dp,M}(j\Omega)$ must approach the periodic repetition of a finite set of ''base-band'' poles and zeros; thus,

$$\lim_{M\to\infty}\tilde{Z}_{dp,M}(j\Omega)=j\tilde{H}\prod_{k=-\infty}^{\infty}\prod_{i=0}^{N-1}\frac{\Omega-\Omega_{2i}+2\pi kF_s}{\Omega-Q_{2i+1}+2\pi kF_s}, \tag{45}$$

where the base-band pole and zero frequencies are defined such that

$$0=\Omega_0<\Omega_1<\Omega_2<\cdots<\Omega_{2N-1}<2\pi F_s\,, \tag{46}$$

$$\Omega_m=2\pi F_s-\Omega_{2N-m}\,,\quad 1\leqslant m\leqslant 2N-1. \tag{47}$$

Equation (45) may be simplified by applying the infinite product expansion of a sine, yielding

$$\tilde{Z}_{dp,g}(j\Omega)=j\tilde{H}\prod_{i=0}^{N-1}\frac{\sin\left(\dfrac{\Omega-\Omega_{2i}}{2F_s}\right)}{\sin\left(\dfrac{\Omega-\Omega_{2i+1}}{2F_s}\right)}. \tag{48}$$

Replacing sine functions by complex exponentials and noting that $z=e^{j\Omega/F_s}$ yields the discrete-time impedance

$$Z_{dp,g}(z)\equiv\tilde{Z}_{dp,g}(j\Omega)=-H\prod_{i=0}^{N-1}\frac{1-z^{-1}e^{j\Omega_{2i}/F_s}}{1-z^{-1}e^{j\Omega_{2i+1}/F_s}}, \tag{49}$$

where, by making use of Eq. (47) H is shown to be

$$H=-j\tilde{H}\exp\left(\frac{j}{2F_s}\sum_{i=0}^{N-1}(\Omega_{2i+1}-\Omega_{2i})\right)=\tilde{H}. \qquad (50)$$

III. POLES AND ZEROS OF THE GLOTTAL DRIVING-POINT IMPEDANCE

Remember that, by applying the series impedance method to the DMI model at $x_N=-Nl$, one obtains a front cavity whose impedance is $Z_{dp,g}(z)$, and a back cavity whose impedance is

$$Z_b(N,z)=Z_g. \qquad (51)$$

The frequencies of the poles and zeros of $Z_{dp,g}(z)$ can be calculated by constructing modified vocal-tract models in which $Z_b(N,z)$ is replaced by values which happen to be more convenient for the problem at hand. Notice, for example, that the zeros of $Z_{dp,g}(z)$ are the roots of the equation

$$0+Z_{dp,g}(z)=0. \qquad (52)$$

Comparing Eqs. (52) and (21), we discover that the zeros of $Z_{dp,g}(z)$ are also the resonances of a modified vocal-tract model in which

$$Z_f(N,z)=Z_{dp,g}(z), \quad Z_b(N,z)=0. \qquad (53)$$

Equation (53) is satisfied by the Q model introduced in Sec. I. The zeros of $Z_{dp,g}(z)$ are therefore the resonances of the Q model, which are given by

$$\Omega_{2i}/F_s=q_i, \qquad (54)$$

where the frequencies q_i are defined in Eq. (29).

In a similar manner, the poles of $Z_{dp,g}(z)$ are also the roots of the equation

$$0+Y_{dp,g}(z)=0, \qquad (55)$$

where $Y_{dp,g}(z)=1/Z_{dp,g}(z)$. Comparing Eqs. (55) and (22) we discover that the poles of $Z_{dp,g}(z)$ are the resonances of a modified vocal-tract model in which

$$Y_f(N,z)=Y_{dp,g}(z), \quad Y_b(N,z)=0. \qquad (56)$$

Equation (56) is satisfied by the R model introduced in Sec. I. The poles of $Z_{dp,g}(z)$ are therefore

$$\Omega_{2i+1}/F_s=r_i, \qquad (57)$$

where r_i are given by Eq. (33).

By combining Eqs. (49), (54), and (57), we obtain

$$Z_{dp,g}(z)=-H\prod_{i=0}^{N-1}\frac{1-e^{jq_i}z^{-1}}{1-e^{jr_i}z^{-1}}=-H\frac{Q(z)}{R(z)}, \qquad (58)$$

which proves the theorem given in Eq. (36).

As a corollary to the main theorem, notice that Eq. (40) dictates that the poles and zeros of $Z_{dp,g}(z)$ must alternate, and that therefore

$$0=q_0<r_0<q_1<\cdots<q_{N-1}<r_{N-1}<2\pi. \qquad (59)$$

Guillemin's analysis (1935) of Foster's reactance theorem shows that the poles and zeros of the driving-point impedance fail to alternate if and only if the circuit contains at least one negative inductor ($L_n<0$) and at least one negative capacitor ($C_n<0$). Negative-valued reactances cannot be realized using passive physical elements—in particular, Eqs. (41) and (42) show that negative-valued reactances correspond to negative cross-sectional areas in the vocal-tract model. Negative reactances can be modeled in a digital simulation, but since the DMI model contains a single positive resistance (the glottal resistance), negative reactances are a sufficient and necessary condition for instability of the circuit. Thus, the well-known interlacing property of the LSFs (Sugamura and Itakura, 1981; Soong and Juang, 1984) may be restated as a corollary of Eq. (36).

A. Corollary (interlacing property of the LSFs)

The following conditions are equivalent:

(a) The LSFs are interlaced [Eq. (59)].
(b) The cross-sectional areas Φ_n in the DMI model are everywhere positive.
(c) The transfer function $T(z)=1/A(z)$ is stable.

IV. CONCLUSIONS

This letter demonstrates that the glottal driving-point impedance of the standard DMI model is proportional to $Q(z)/R(z)$, where $Q(z)$ and $R(z)$ are the standard LSF polynomials.

In this way, a new definition of the LSFs is proposed: the LSFs are the poles and zeros of the glottal driving-point impedance of a discrete matched-impedance vocal-tract model. This new definition is more concise than any previous definition. The new definition also allows practitioners to think about the LSFs using a type of physical reasoning which was not previously possible. For example, Sec. III shows that the well-known interlacing property of the LSFs may be proven using a circuit theorem known as Foster's reactance theorem, which is frequently applied in acoustic phonetic analysis.

Foster, R. M. (**1924**). ''A reactance theorem,'' Bell Syst. Tech. J. ■■, 259–267.

Fujimura, O. (**1962**). ''Analysis of nasal consonants,'' J. Acoust. Soc. Am. **34**, 1865–1875.

Furui, S. (**1989**). *Digital Speech Processing, Synthesis, and Recognition* (Dekker, New York).

Guillemin, E. A. (**1935**). *Communication Networks* (Wiley, New York), Vol. II.

O'Shaughnessy, D. (**1987**). *Speech Communication* (Addison-Wesley, Reading, MA).

Soong, F., and Juang, B.-H. (**1984**). ''Line spectral pair (LSP) and speech data compression,'' in Proceedings of the ICASSP, pp. 1.10.1–1.10.4 (unpublished).

Sugamura, N., and Itakura, F. (**1981**). ''Speech data compression by LSP speech analysis–synthesis technique,'' Trans. IECE **J 64-A**(8), 599–606 (in Japanese).