PRACTICE EXAM 2

Tuesday, October 22, 2019

• This is a PRACTICE exam. On the real exam, you’ll be allowed to use one sheet (front and back) of handwritten notes.

• No calculators are permitted. You need not simplify explicit numerical expressions.

• The real exam will have a total of 50 points. Each problem specifies its point total. Plan your work accordingly.

• You must SHOW YOUR WORK to get full credit.
Possibly Useful Formulas

YPbPr and Sobel Mask

\[
\begin{bmatrix}
Y \\
P_b \\
P_r
\end{bmatrix}
= \begin{bmatrix}
0.299 & 0.587 & 0.114 \\
-0.168736 & -0.331264 & 0.5 \\
0.5 & -0.418688 & -0.081312
\end{bmatrix}
\begin{bmatrix}
R \\
G \\
B
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
2 & 0 & -2 \\
1 & 0 & -1
\end{bmatrix}
\ast \ast I[n_1, n_2], \quad
\begin{bmatrix}
1 & 2 & 1 \\
0 & 0 & 0 \\
-1 & -2 & -1
\end{bmatrix}
\ast \ast I[n_1, n_2]
\]

Integral Image and Lowpass Filter

\[
i[i, j] = \sum_{m_1=0}^{i} \sum_{m_2=0}^{j} i[m_1, m_2]
\]

\[
H(\omega_1, \omega_2) = \begin{cases}
1 & |\omega_1| < \phi_1, \quad |\omega_2| < \phi_2 \\
0 & \text{otherwise}
\end{cases}
\quad
h[x_1, x_2] = \left(\frac{\phi_1}{\pi}\right) \left(\frac{\phi_2}{\pi}\right) \text{sinc}(\phi_1 x_1) \text{sinc}(\phi_2 x_2)
\]

Orthogonality Principle and LPC

\[
\varepsilon = E \left[ \left( x[n] - \sum_{m=1}^{P} \alpha_m x[n-m] \right)^2 \right], \quad
\frac{\partial \varepsilon}{\partial \alpha_k} = -2 E \left[ x[n-k] \left( x[n] - \sum_{m=1}^{P} \alpha_m x[n-m] \right) \right]
\]

\[
R_{xx}[k] = \sum_{m=1}^{12} \alpha_m R_{xx}[k-m]
\]

Autocorrelation and Power Spectrum

\[
R_{xx}[n] = E \{ x[m] x[m-n] \} \leftrightarrow S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n] e^{-j\omega n}
\]

\[
r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m] x[m-n] \leftrightarrow s_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[n] e^{-j\omega n}
\]

Fourier Series

\[
x[n] = \sum_{k=0}^{P-1} X_k e^{j2\pi kn/P}, \quad X_k = \frac{1}{P} \sum_{n=0}^{P-1} x[n] e^{-j2\pi kn/P}
\]

Autocorrelation and Power Spectrum

\[
R_{xx}[n] = E \{ x[m] x[m-n] \} \leftrightarrow S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n] e^{-j\omega n}
\]

\[
r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m] x[m-n] \leftrightarrow s_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[n] e^{-j\omega n}
\]
Problem 1 (30 points)

Suppose you have an RGB image \(i[n_1, n_2, n_3]\) with \(0 \leq n_1 < N_1\) rows, \(0 \leq n_2 < N_2\) columns, and \(0 \leq n_3 < 3\) color planes. The matched filter in part (d) of this problem is of size \(M_1 \times M_2\). Use big-\(O\) notation, in terms of the variables \(N_1, N_2, M_1\) and \(M_2\), to express the complexity of each of the following operations:

(a) Converting from RGB to YPbPr color space.
(b) Computing the horizontal and vertical gradients of each color plane using a Sobel mask.
(c) Lowpass filtering (after zero-padding, so that the output is of the same size, \(N_1 \times N_2 \times 3\), as the input) with a separable ideal anti-aliasing filter whose frequency response is

\[
H(\omega_1, \omega_2) = \begin{cases} 
1 & |\omega_1| < \frac{\pi}{3}, \ |\omega_2| < \frac{\pi}{3} \\
0 & \text{otherwise}
\end{cases}
\]

(d) Filtering with a matched filter of size \(M_1\) rows, \(M_2\) columns.
(e) Calculating the integral image \(ii[n_1, n_2] = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} i[m_1, m_2, 0]\) for all \(0 \leq n_1 < N_1\) and \(0 \leq n_2 < N_2\).
(f) Given the integral image, find the box-summation \(f[b_1, b_2, e_1, e_2]\) defined as

\[
f[b_1, b_2, e_1, e_2] = \sum_{n_1=b_1}^{e_1} \sum_{n_2=b_2}^{e_2} i[n_1, n_2, 0]
\]

for all values \(0 \leq b_1 \leq N_1 - 1, 0 \leq e_1 \leq N_1 - 1, 0 \leq b_2 \leq N_2 - 1, 0 \leq e_2 \leq N_2 - 1\).

Problem 2 (10 points)

Suppose you have an input image with 8-bit integer pixel values, \(0 \leq i[n_1, n_2, n_3] \leq 255\), where \(n_1\) is the row index, \(n_2\) is the column index, and \(n_3\) is the color plane. What are the minimum and maximum pixel values that result as the outputs of the following operations:

(a) Convert to a YPbPr color space. What are the minimum and maximum possible values of \(Y\), \(P_b\), and \(P_r\)?
(b) Compute the horizontal and vertical gradients using a Sobel mask. What are the minimum and maximum possible values of each of the two gradient images?

Problem 3 (5 points)

Consider the infinite-sized image \(i[n_1, n_2] = \delta[n_1 - 5]\), i.e.,

\[
i[n_1, n_2] = \begin{cases} 
1 & n_1 = 5 \\
0 & \text{otherwise}
\end{cases}
\]

Use a Sobel mask to find the resulting images \(G_x[n_1, n_2]\) and \(G_y[n_1, n_2]\).
Problem 4  (5 points)

Suppose you want to find the horizon line in a grayscale image \( i[n_1, n_2] \). Suppose the horizon line is defined to be the row index \( n_1 \) that maximizes the brightness difference \( BD[n_1] \), defined as

\[
BD[n_1] = \sum_{m_2=0}^{N_2-1} \left( \frac{1}{n_1} \sum_{m_1=0}^{n_1-1} i[m_1, m_2] \right) - \left( \frac{1}{N_1-n_1} \sum_{m_1=n_1}^{N_1-1} i[m_1, m_2] \right)
\]

You are given the integral image \( ii[n_1, n_2] = \sum_{n_1=0}^{n_1} \sum_{n_2=0}^{n_2} i[m_1, m_2] \). Devise a formula that uses \( ii[n_1, n_2] \) to compute \( BD[n_1] \) with a small constant number of operations per candidate horizon line.

Problem 5  (5 points)

Consider the problem of upsampling, by a factor of 2, the infinite-sized image

\[
x[n_1, n_2] = \delta[n_1 - 5] = \begin{cases} 
1 & n_1 = 5 \\
0 & \text{otherwise}
\end{cases}
\]

Suppose that the image is upsampled, then filtered, as

\[
y[n_1, n_2] = \begin{cases} 
x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\
0 & \text{otherwise}
\end{cases} \\
z[n_1, n_2] = y[n_1, n_2] * h[n_1, n_2]
\]

Let \( h[n_1, n_2] \) be the ideal anti-aliasing filter with frequency response

\[
H(\omega_1, \omega_2) = \begin{cases} 
1 & |\omega_1| < \frac{\pi}{2}, \ |\omega_2| < \frac{\pi}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Find \( z[n_1, n_2] \).

Problem 6  (5 points)

The stochastic autocorrelation of a periodic signal is periodic, \( R_{xx}[P] = R_{xx}[0] \). How about the signal autocorrelation? Suppose that the frame length is an integer multiple of the number of periods, \( L = kP \), so that

\[
x[n] = \begin{cases} 
\text{periodic with period } P & 0 \leq n \leq kP - 1 \\
0 & \text{otherwise}
\end{cases}
\]

Find \( r_{xx}[P] \) in terms of \( r_{xx}[0] \).

Problem 7  (10 points)

Consider the signal \( x[n] = \beta^n u[n] \), where \( u[n] \) is the unit step function.

(a) Find the LPC coefficient, \( \alpha \), that minimizes \( \varepsilon \), where

\[
\varepsilon = \sum_{n=-\infty}^{\infty} e^2[n], \quad e[n] = x[n] - \alpha x[n-1]
\]
(b) Find the signal $e[n]$ that results from your choice of $\alpha$ in part (a).

Problem 8  (10 points)

Consider the LPC synthesis filter $s[n] = e[n] + \alpha s[n - 1]$.

(a) Under what condition on $\alpha$ is the synthesis filter stable?

(b) Assume that the synthesis filter is stable. Suppose that $e[n]$ is the pulse train $e[n] = \sum_{p=-\infty}^{\infty} \delta[n - pP]$. As a function of $\alpha$, $P$, and $\omega$, what is the DTFT $S(e^{j\omega})$? You need not simplify, but your answer should contain no integrals or infinite sums.