1. DTFT Review

2. Windowing

3. Practical Windows
Outline

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When two signals are convolved, their DTFTs get multiplied together

\[ y[n] = h[n] \ast x[n] \iff Y(\omega) = H(\omega)X(\omega) \]

where

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega \]
Ideal Filters

Probably the most important DTFT pairs are the ideal LPF, BPF, and HPF:

\[
H_{LPF}(\omega) = \begin{cases} 
  1 & |\omega| < \omega_c \\
  0 & \text{else}
\end{cases} \quad \Leftrightarrow \quad h_{LPF}[n] = \begin{cases} 
  \frac{\sin(\omega_c n)}{\pi n} & n \neq 0 \\
  \frac{\omega_c}{\pi} & n = 0
\end{cases}
\]

\[
H_{BPF}(\omega) = \begin{cases} 
  1 & \omega_1 < |\omega| < \omega_2 \\
  0 & \text{else}
\end{cases} \quad \Leftrightarrow \quad h_{BPF}[n] = \begin{cases} 
  \frac{\sin(\omega_2 n) - \sin(\omega_1 n)}{\pi n} & n \neq 0 \\
  \frac{\omega_2 - \omega_1}{\pi} & n = 0
\end{cases}
\]

\[
H_{HPF}(\omega) = \begin{cases} 
  1 & \omega_1 < |\omega| \leq \pi \\
  0 & \text{else}
\end{cases} \quad \Leftrightarrow \quad h_{HPF}[n] = \begin{cases} 
  \delta[n] - \frac{\sin(\omega_1 n)}{\pi n} & n \neq 0 \\
  1 - \frac{\omega_1}{\pi} & n = 0
\end{cases}
\]
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Real filters are usually FIR

The problem with ideal filters is that

\[ h[n] = \frac{\sin(\omega_c n)}{\pi n} = \left(\frac{\omega_c}{\pi}\right) \text{sinc}(\omega_c n) \]

is infinite length. Infinite length convolution is not possible on a computer with finite memory and finite time:

\[ y[n] = \sum_{m=-\infty}^{\infty} h[n]x[n - m] \]

A real-world filter needs to have finite computation. For that, it usually needs a **finite impulse response** (FIR), e.g., a filter of length \( N = 2M + 1 \) requires \( N \) multiply operations per output sample:

\[ y[n] = \sum_{m=-M}^{M} h[m]x[n - m] \]
Frequency Sampling Doesn’t Work

You’re probably thinking: why FIR? Why not just do it all in the frequency domain?

\[ x[n] \rightarrow X(\omega) \rightarrow Y(\omega) = H(\omega)X(\omega) \rightarrow y[n] \quad (1) \]

The problem is that we’d need to do this with discrete omega, \( \omega = \frac{2\pi k}{N} \), instead of continuous \( \omega \). The only way a computer can do this is using, effectively, a discrete Fourier series:

\[ x[n] \rightarrow X_k \rightarrow Y_k = H \left( \frac{2\pi k}{N} \right) X_k \rightarrow y[n] \quad (2) \]

The problem is that Eq. 2 gives a different result from Eq. 1. Eq. 2 pretends that \( x[n] \) is periodic, with a period of \( N \) samples, even if it isn’t really. The pretense of periodicity causes artifacts: multiplying by \( H(\omega) \) causes the imagined periods of \( x[n] \) to blur into one another. This is called temporal aliasing.
FIR filters need to be windowed

So we need to do it all in the time domain. This is exactly what numpy.convolve does:

\[ y[n] = \sum_{m=-M}^{M} h[m][n - m] \]

The problem is that the FIR and IIR (infinite impulse response) filters are not the same. In fact, they are related by windowing:

\[ h_{FIR}[m] = w[m]h_{IIR}[m] = \begin{cases} h_{IIR}[m] & |m| \leq M \\ 0 & \text{else} \end{cases} \]

\[ w[m] = \begin{cases} 1 & |m| \leq M \\ 0 & \text{otherwise} \end{cases} \]
The Result of Windowing

What does windowing do? To get a hint, remember the convolution property of DTFT:

\[ y[n] = h[n] * x[n] \Leftrightarrow Y(\omega) = H(\omega)X(\omega) \]

It turns out that almost the same thing works in reverse:

\[ h_{FIR}[n] = w[n]h_{IIR}[n] \Leftrightarrow H_{FIR}(\omega) = \frac{1}{2\pi} W(\omega) * H_{IIR}(\omega) \]

Now we need to define “convolution in frequency.” We define it like this:

\[ H_{FIR}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\theta)H_{IIR}(\omega - \theta) d\theta \]
Convolution in Frequency Example: IIR

As an easy-to-compute example, suppose

\[ h[n] = \left( \frac{\omega_c}{\pi} \right) \text{sinc}(\omega_c n) \leftrightarrow H(\omega) = \begin{cases} 
1 & |\omega| < \omega_c \\
0 & \text{otherwise}
\end{cases} \]

Suppose we square it:

\[ g[n] = h^2[n] = \left( \frac{\omega_c}{\pi} \right)^2 \text{sinc}^2(\omega_c n) \]

Then

\[ G(\omega) = \frac{1}{2\pi} H(\omega) \ast H(\omega) = \begin{cases} 
\left( \frac{\omega_c - |\omega|}{\pi} \right) & |\omega| < 2\omega_c \\
0 & \text{otherwise}
\end{cases} \]
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The most useful window is the rectangular window:

\[
\begin{align*}
    w_R[n] &= \begin{cases} 
        1 & |n| \leq M \\
        0 & \text{else}
    \end{cases} \\
    W_R(\omega) &= \frac{\sin(\omega N/2)}{\sin(\omega/2)}
\end{align*}
\]

where \( N = 2M + 1 \) is the length of the window.
Windowing with a Rectangular Window

Window with a rectangular window:

\[ h_{FIR}[n] = w_R[n]h_{IIR}[n] \leftrightarrow H_{FIR}(\omega) = \frac{1}{2\pi} W_R(\omega) \ast H_{IIR}(\omega) \]

Causes the following effects:

- \( H_{FIR}(\omega) \) has **ripples in the passband**, going up and down, crossing the value \( H_{FIR}(\omega) = 1 \) only once every \( 2\pi/N \) radians.
- \( H_{FIR}(\omega) \) has **ripples in the stopband**, going up and down, crossing the value \( H_{FIR}(\omega) = 0 \) only once every \( 2\pi/N \) radians.
- \( H_{FIR}(\omega) \) has a gradual **transition band** between the passband and stopband, with a width of \( 2\pi/N \).
Other Useful Windows

- Sidelobes can be made smaller, at the expense of a wider transition band.
- This is done by making the window less abrupt in the time domain.
Other Useful Windows

Triangular (Bartlett): \( w_B[n] = \left( 1 - \frac{|n|}{M + 1} \right) w_R[n] \)

Hamming Window: \( w_H[n] = \left( 0.54 + 0.46 \cos \left( \frac{\pi n}{M} \right) \right) w_R[n] \)

Hann Window: \( w_N[n] = \left( 0.5 + 0.5 \cos \left( \frac{\pi n}{M} \right) \right) w_R[n] \)
Other Useful Windows

<table>
<thead>
<tr>
<th>Window</th>
<th>First Null</th>
<th>First Sidelobe</th>
<th>Other Sidelobes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$\omega = \frac{2\pi}{N}$</td>
<td>-13dB</td>
<td>$1/N$</td>
</tr>
<tr>
<td>Triangular</td>
<td>$\omega = \frac{4\pi}{N}$</td>
<td>-26dB</td>
<td>$1/N^2$</td>
</tr>
<tr>
<td>Hamming</td>
<td>$\omega = \frac{4\pi}{N}$</td>
<td>-44dB</td>
<td>flat</td>
</tr>
<tr>
<td>Hann</td>
<td>$\omega = \frac{4\pi}{N}$</td>
<td>-32dB</td>
<td>Very Small</td>
</tr>
</tbody>
</table>