Lecture 12: Sampled Systems

ECE 401: Signal and Image Analysis

University of Illinois

3/7/2017
1 Frequency Response Review
2 Sampled Systems
3 Anti-Aliasing
4 Sampling
5 Filtering
6 Ideal D/A
Find the frequency response of this system. Express it as

\[ H(\omega) = A(\omega)e^{j\theta(\omega)} \]

where \( A(\omega) \) is something real-valued.

\[ h[n] = \begin{cases} 
1 & n = 0 \\
-1 & n = 2 \\
0 & \text{otherwise}
\end{cases} \]
Computers can be used to generate signal $y(t)$, given some input signal $x(t)$. The procedure is:

1. Filter $x(t)$ through an anti-aliasing filter, giving the filtered signal $\tilde{x}(t)$.
2. Sample $\tilde{x}(t)$ at a sampling rate of $F_S$, giving digital signal $x[n]$.
4. Pass $y[n]$ through a D/A to generate $y(t)$. 
Outline

1. Frequency Response Review
2. Sampled Systems
3. Anti-Aliasing
4. Sampling
5. Filtering
6. Ideal D/A
Anti-Aliasing Filter

- $x(t) = \cos(\Omega_1 t)$ gets sampled as $x[n] = \cos \left( \left( \frac{\Omega_1}{F_s} \right) n \right)$, which can equivalently be written as $x[n] = \cos \left( \left( \frac{\Omega_1}{F_s} - 2\pi\ell \right) n \right)$ for any integer value of $\ell$.

- If $x[n]$ is then passed back immediately through an ideal D/A, it will only equal $x(t)$ if $|\Omega_1| < \pi F_s$.

- Otherwise, it will be “aliased” to a new frequency, $\Omega_2 = \Omega_1 - 2\pi\ell F_s$, such that $|\Omega_2| < \pi F_s$.

- We can avoid aliasing by first removing, from $x(t)$, any stuff that would get aliased. This is done using a continuous-time lowpass filter:

$$x(t) \rightarrow H_{LPF}(\Omega) \rightarrow \tilde{x}(t)$$

where

$$H_{LPF}(\Omega) = \begin{cases} 
1 & |\Omega| < \pi F_s \\
0 & \text{otherwise}
\end{cases}$$
Example: Fourier Series

Suppose $x(t)$ is periodic with period $T_0$, so that

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

Then after we filter it with the anti-aliasing filter, we get

$$\tilde{x}(t) = \sum_{k=-K}^{K} X_k e^{jk\Omega_0 t}$$

where $K$ is the largest integer such that

$$K\Omega_0 < \pi F_s$$

Note $\tilde{x}(t) \neq x(t)$! The high-frequency harmonics (the harmonics that would get aliased by sampling) have been removed; they are gone forever. If you want to keep them, you need to use a higher sampling rate.
Outline

1. Frequency Response Review
2. Sampled Systems
3. Anti-Aliasing
4. Sampling
5. Filtering
6. Ideal D/A
We sample by measuring the signal $F_s$ times/second:

$$x[n] = \tilde{x}\left(t = \frac{n}{F_s}\right)$$

... so if ...

$$\tilde{x}(t) = \sum_{k=-K}^{K} X_k e^{jk\Omega_0 t}$$

then

$$x[n] = \sum_{k=-K}^{K} X_k e^{jk\omega_0 n}, \quad k\omega_0 = \frac{k\Omega_0}{F_s}$$
Outline

1. Frequency Response Review
2. Sampled Systems
3. Anti-Aliasing
4. Sampling
5. Filtering
6. Ideal D/A
Now we implement any processing we want, in discrete time. For example, if we have an LTI system:

\[ y[n] = h[n] * x[n] \]

and

\[ x[n] = \sum_{k=-K}^{K} X_k e^{j \omega_0 n} \]

then

\[ y[n] = \sum_{k=-K}^{K} H(k \omega_0) X_k e^{j k \omega_0 n} \]

where \( H(k \omega_0) \) is the frequency response, \( H(\omega) \), evaluated at the frequency of the \( k^{th} \) harmonic, which is \( \omega = k \omega_0 \).
Example: Averager

For example, suppose we average $N$ consecutive samples:

$$y[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[n - m]$$

The frequency response is

$$H(\omega) = e^{-j\omega \left( \frac{N-1}{2} \right)} \frac{\sin(\omega N/2)}{N \sin(\omega/2)}$$

$$= \begin{cases} 
1 & \omega = 0 \\
0 & \omega = \frac{2\pi \ell}{N}, \ 0 < \text{integer } \ell < N \\
\text{other values} & \text{other frequencies}
\end{cases}$$
Example: Averager

So the output signal is

\[ y[n] = \sum_{k=-K}^{K} Y_k e^{j k \omega_0 n} \]

where

\[ Y_k = e^{-j k \omega_0 \left( \frac{N-1}{2} \right)} \frac{\sin(k \omega_0 N/2)}{N \sin(k \omega_0 /2)} X_k \]

For example

- \( Y_0 = X_0 \) exactly
- \( Y_k = 0 \) when \( k \omega_0 = \frac{2 \pi \ell}{N} \) for nonzero integers \( \ell < N \).
Outline

1. Frequency Response Review
2. Sampled Systems
3. Anti-Aliasing
4. Sampling
5. Filtering
6. Ideal D/A
Now we send the generated signal through an ideal D/A. If $x(t)$ was periodic, and if the digital processing was LTI, then the output will also be periodic with the same period:

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$$

The relationship between $X_k$ and $Y_k$ is determined by two things: (1) the anti-aliasing filter and D/A, and (2) the frequency response.
Effect of the anti-aliasing filter and ideal D/A: an ideal D/A can only generate signal up to frequencies of $\Omega = \pi F_s$, so it limits $Y_k$ as follows:

$$Y_k = 0 \text{ for all } |k\Omega_0| \geq \pi F_s$$

Effect of the digital filter: For frequencies $|\Omega| < \pi F_s$, the following dimensional analysis works:

$$\left(\omega \text{ radians/sample}\right) \times \left(F_s \text{ samples/second}\right) = \left(\Omega \text{ radians/second}\right)$$

Putting it all together,

$$Y_k = \begin{cases} H \left(\frac{k\Omega_0}{F_s}\right) X_k & |k\Omega_0| < \pi F_s \\ 0 & |k\Omega_0| \geq \pi F_s \end{cases}$$
Example: Averager

For example, suppose

\[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j k \Omega_0 t} \]

\[ x(t) \rightarrow H_{LPF}(\Omega) \rightarrow \tilde{x}(t) \]

\[ x[n] = \tilde{x}\left(t = \frac{n}{F_s}\right) \]

\[ y[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[n - m] \]
Example: Averager

Then \( y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j k \Omega_0 t} \), where

- \( Y_k = 0 \) for \( |k \Omega_0| > \pi F_s \)
- For other \( k \),

\[
Y_k = e^{-j \frac{k \Omega_0}{F_s} \left( \frac{N-1}{2} \right) \sin \left( \frac{k \Omega_0 N}{2 F_s} \right)} X_k
\]

In particular:

- \( Y_0 = X_0 \), they have the same DC offset
- \( Y_k = 0 \) for any \( k \) such that \( k \Omega_0 \) is a multiple of \( 2 \pi F_s / N \).

Another way to say the same thing: \( Y_k = 0 \) if \( k F_0 \) is a multiple of \( F_s / N \). It’s like the averager has laid down a list of zeros, which knock out any harmonics that happen to land at integer multiples of \( F_s / N \).