Lecture 11: Frequency Response

ECE 401: Signal and Image Analysis

University of Illinois

3/2/2017
1. LTI Review

2. Frequency Response

3. Frequency Response of an Averager
Outline

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3. Frequency Response of an Averager
Is this system linear? Is it time-invariant? Can you prove your answers?

\[ y[n] = x[n] + x[n + 5] \]
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Frequency Response Example

Consider the system

\[ y[n] = x[n - 1] + x[n] + x[n + 1] \]

Suppose the input is a cosine at some frequency \( \omega \),
\( x[n] = \cos(\omega n) \). Then the output is

\[ y[n] = \cos(\omega (n - 1)) + \cos(\omega n) + \cos(\omega (n + 1)) \]

Using the phasor method, we can write this as

\[ y[n] = \Re \{ e^{j \omega n} e^{-j \omega} + e^{j \omega n} + e^{j \omega n} e^{j \omega} \} \]
\[ = \Re \{ (e^{-j \omega} + 1 + e^{j \omega}) e^{j \omega n} \} \]
\[ = \Re \{ (1 + 2 \cos(\omega)) e^{j \omega n} \} = (1 + 2 \cos(\omega)) \cos(\omega n) \]

So the output is a cosine at **exactly the same frequency**, but scaled by the frequency-dependent scaling factor

\[ H(\omega) = 1 + 2 \cos(\omega) \]
Consider the LTI system

\[ y[n] = x[n] \ast h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n - m] \]

Suppose the input is \( x[n] = e^{j\omega n} \). Then the output is

\[
y[n] = \sum_{m=-\infty}^{\infty} h[m]e^{j\omega(n-m)} = e^{j\omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}
\]

\[ = e^{j\omega n}H(\omega) \]

So the output is a complex exponential at exactly the same frequency, but scaled by the complex-valued, frequency-dependent constant

\[ H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \]
Frequency Response Definition

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m}$$

$x[n] = \text{Complex Exponential}$

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)x[n]$$
Frequency Response: Sinusoidal Inputs

\( x[n] = \text{Cosine} \)

\[ x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega)) \]

\( x[n] = \text{Sine} \)

\[ x[n] = \sin(\omega n) \rightarrow y[n] = |H(\omega)| \sin(\omega n + \angle H(\omega)) \]

where \( |H(\omega)| \) and \( \angle H(\omega) \) are just the magnitude and phase of \( H(\omega) \), i.e.,

\[ H(\omega) = |H(\omega)| e^{j\angle H(\omega)} \]
Example: Averager = The Simplest Lowpass Filter

\[ h[n] = \delta[n] + \delta[n - 1] \]

\[
H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = 1 + e^{-j\omega} \\
= e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) = 2e^{-j\omega/2}\cos(\omega/2)
\]

So

\[
|H(\omega)| = 2\cos(\omega/2), \quad \angle H(\omega) = -\omega/2
\]

Notice that \( H(0) = 1 \), while \( H(\pi) = 0 \), so this is a *lowpass filter*. Thus if \( x[n] = \cos(\omega n) \) then

\[
y[n] = \left(2\cos\left(\frac{\omega}{2}\right)\right)\cos\left(\omega \left(n - \frac{1}{2}\right)\right) = \begin{cases} 2\cos(\omega(n - 1/2)) & \omega = 0 \\ 0 & \omega = \pi \end{cases}
\]
Example: Euler Differencer = The Simplest Highpass Filter

\[ h[n] = \delta[n] - \delta[n-1] \]

\[ H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = 1 - e^{-j\omega} \]

\[ = e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) = 2je^{-j\omega/2}\sin(\omega/2) \]

So

\[ |H(\omega)| = 2\sin(\omega/2), \quad \angle H(\omega) = \frac{\pi - \omega}{2} \]

Notice that \( H(0) = 0 \), while \( H(\pi) = 1 \), so this is a highpass filter. Thus if \( x[n] = \cos(\omega n) \) then

\[ y[n] = \left(2\sin\left(\frac{\omega}{2}\right)\right)\cos\left(\omega \left(n - \frac{1}{2}\right) + \frac{\pi}{2}\right) = \begin{cases} 0 & \omega = 0 \\ 2\sin(\omega(n - 1/2)) & \omega = \pi \end{cases} \]
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Frequency Response of an Averager, in General

\[ h[n] = u[n] - u[n - N] \quad \text{for some integer } N \]

\[ H(\omega) = \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} = \sum_{m=0}^{N-1} e^{-j\omega m} \]

In order to solve this one, we need to use Zeno’s paradox, which can be stated as follows. For any fraction \( a \) such that \( |a| < 1 \),

\[ \sum_{m=0}^{\infty} a^m = \frac{1}{1 - a} \]

(In the fable created by the ancient Greek philosopher Zeno of Elea, the fraction is \( a = \frac{1}{2} \)).
\[ H(\omega) = \sum_{m=0}^{\infty} e^{-j\omega m} - \sum_{m=N}^{\infty} e^{-j\omega m} \]

\[ = \sum_{m=0}^{\infty} e^{-j\omega m} - e^{-j\omega N} \sum_{m=0}^{\infty} e^{-j\omega m} \]

using Zeno’s paradox, we convert this to

\[ = \frac{1}{1 - e^{-j\omega}} - \frac{e^{-j\omega N}}{1 - e^{-j\omega}} \]

\[ = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \left( \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \right) \left( \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \]

\[ = e^{-j\omega \frac{(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \]
So the frequency response of this averager:

\[ h[n] = u[n] - u[n - N] \quad \text{for some integer } N \]

is

\[ H(\omega) = A(\omega)e^{-j\theta(\omega)} \]

where

\[ A(\omega) = \left( \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right) \quad \theta(\omega) = -\omega(N - 1)/2 \]
The **signed-amplitude response** of an averager has the following important characteristics

\[ A(\omega) = \left( \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right) \]

We call that the **signed-amplitude response** because it can be either positive or negative; we only require that it should be real. So it’s not exactly the same thing as the magnitude of the complex number.

\[ A(\omega) = \begin{cases} 
N & \omega = 0 \\
0 & \omega = 2\pi \ell/N, \text{ any integer } \ell \neq 0 
\end{cases} \]

In particular, \( H(\pi) = 0 \), so this is a lowpass filter. We could say that the \( N \)-point averager is much more lowpass than the 2-point averager; its **cutoff frequency** is \( \omega = 2\pi/N \).
The **phase response** of an averager has the following important characteristic:

\[ \theta(\omega) = -\omega(N - 1)/2 \]

Notice that this phase is a **linear** function of \( \omega \) (we say the filter has **generalized linear phase**). In general, a linear phase filter is one whose phase response looks like

\[ \theta(\omega) = -\omega d \]

for any constant \( d \). The constant \( d \) is called the **filter delay**, because

\[ x[n] = \cos(\omega n) \rightarrow y[n] = A(\omega) \cos(\omega (n - d)) \]

So the filter acts as though it **delays** the input, by a delay of \( d \) samples.