For exam 2, you need to know (1) impulse response, (2) linearity and time invariance, (3) frequency response, (4) discrete-time processing of continuous-time periodic signals, (5) DTFT. In other words, you should be able to do the following problems.

2.1 Impulse Response

Problem:
A particular system computes the average of 5 samples, minus the average of the previous five:

\[ y[n] = \frac{1}{5} \sum_{m=0}^{4} x[n-m] - \frac{1}{5} \sum_{m=5}^{9} x[n-m] \]

Find the impulse response \( h[n] \).

Solution:
Feed \( \delta[n] \) into the system, and the output is:

\[ h[n] = \begin{cases} 
\frac{1}{5} & 0 \leq n \leq 4 \\
-\frac{1}{5} & 5 \leq n \leq 9 \\
0 & \text{otherwise}
\end{cases} \]

2.2 Linearity and Time Invariance

Problem:
A particular system changes the sign of every second input sample, thus

\[ y[n] = (-1)^n x[n] \]


Problem:

\[ x_1[n] \rightarrow (-1)^n x_1[n] \]
\[ x_2[n] \rightarrow (-1)^n x_2[n] \]
\[ ax_1[n] + bx_2[n] \rightarrow (-1)^n (ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n] \]

So the system is linear.

\[ x_1[n-m] = (-1)^{n-m} x_1[n-m] \]
\[ x_1[n-m] \rightarrow (-1)^n x_1[n-m] \neq y_1[n-m] \]

So the system is not time-invariant.

2.3 DT Processing of CT Signals

Problem:
A continuous-time periodic signal \( x(t) \) with a period of \( T_0 = 0.03 \) seconds is passed through an ideal anti-aliasing filter \( H_a(\Omega) \), then sampled at \( F_s = 1000 \) Hz to produce the discrete-time signal \( x[n] \). The discrete-time signal \( x[n] \) is then processed by the following system:

\[ y[n] = \frac{1}{10} \sum_{m=0}^{9} x[n-m] \]

The signal \( y[n] \) is then passed through an ideal D/A, at the same sampling frequency \( F_s = 1000 \) Hz, to produce the signal \( y(t) \). Let \( Y_k \) be the Fourier series coefficients of \( y(t) \), and \( X_k \) those of \( x(t) \).
1. Assume that $X_k \neq 0$ for all $k$; identify all $k$ for which $Y_k = 0$.

2. Specify $|Y_k|$ in terms of $|X_k|$ for all non-zero $Y_k$.

**Solution:**

The harmonic frequencies of both $x(t)$ and $y(t)$ are $k\Omega_0 = 2\pi k/T_0 = 200\pi/3$ radians/second. The anti-aliasing filter eliminates all components $|\Omega| \geq \pi F_s = 1000\pi$, thus, all harmonics with $|k| \geq 15$. The other harmonics are mapped to $k\omega_0 = k\Omega_0/F_s = 2\pi k/30$. The discrete-time system has the frequency response

$$H(\omega) = e^{-j\omega(10-1)/2} \frac{\sin(\omega 10/2)}{10 \sin(\omega/2)}$$

which has zeros at $\omega = 2\pi \ell/10$ for $\ell \neq 0$, thus every third harmonic of $x[n]$ is zeroed out. Thus

$$|Y_k| = \begin{cases} 0 & k = 3\ell, \ \ell \neq 0 \\ 0 & |k| \geq 15 \\ \left| \frac{\sqrt{3}}{20 \sin(k\pi/30)} X_k \right| & \text{otherwise} \end{cases}$$

### 2.4 DTFT

**Problem:**

A particular bandpass filter has the frequency response

$$F(\omega) = \begin{cases} 1 & \frac{\pi}{10} \leq |\omega| \leq \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response $f[n]$.

**Solution:**

If your formula sheet has the form of an ideal lowpass filter on it, then the easiest way to solve this problem is to notice that

$$F(\omega) = G(\omega) - H(\omega)$$

where $G(\omega)$ is an ideal LPF with a cutoff of $\pi/4$, and $H(\omega)$ is an ideal LPF with a cutoff of $\pi/10$, thus

$$f[n] = \frac{\sin(\pi n/4) - \sin(\pi n/10)}{\pi n}$$

Alternatively, you could use the inverse DTFT formula directly:

$$f[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\pi/10}^{\pi/4} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi jn} \left[ e^{j\pi n/4} - e^{j\pi n/10} + e^{-j\pi n/10} - e^{-j\pi n/4} \right] = \frac{\sin(\pi n/4) - \sin(\pi n/10)}{\pi n}$$